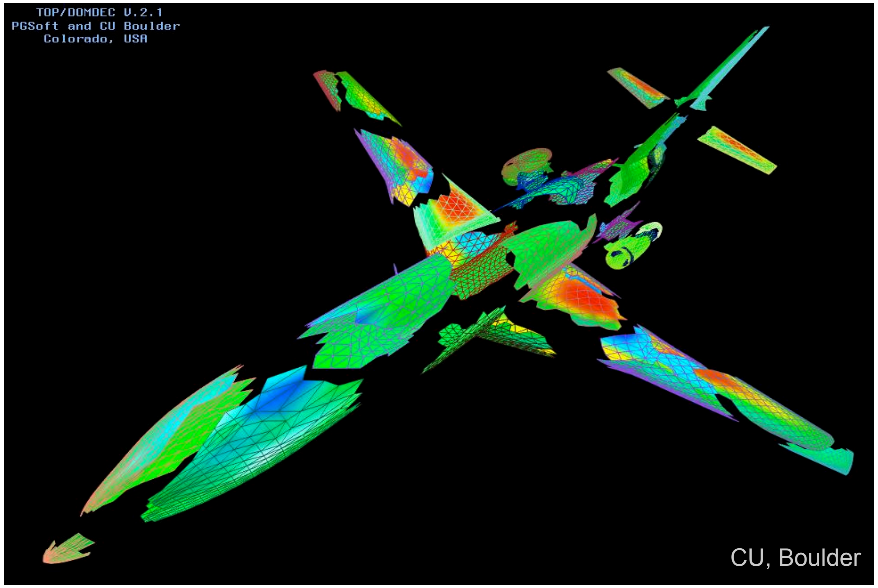
Dynamic Substructuring Concepts

Tutorial



IMAC 2010

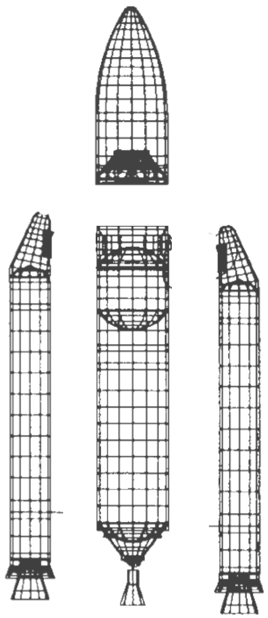
TU Delft
Delft University of Technology

Daniel Rixen
Delft University of Technology,
The Netherlands

Substructuring is a “**way to see things in parts**” in order

- to simplify the dynamic analysis

- to concentrate on specific components



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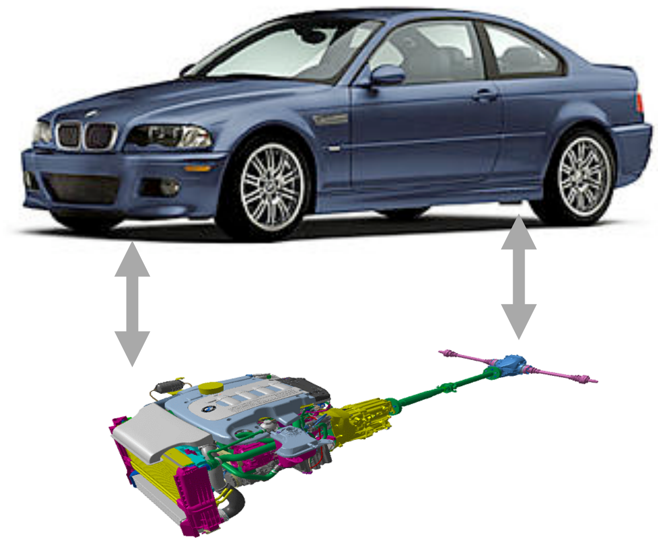

Substructuring ideas are already more then 40 years old … but still a research field.

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General concepts in substructuring

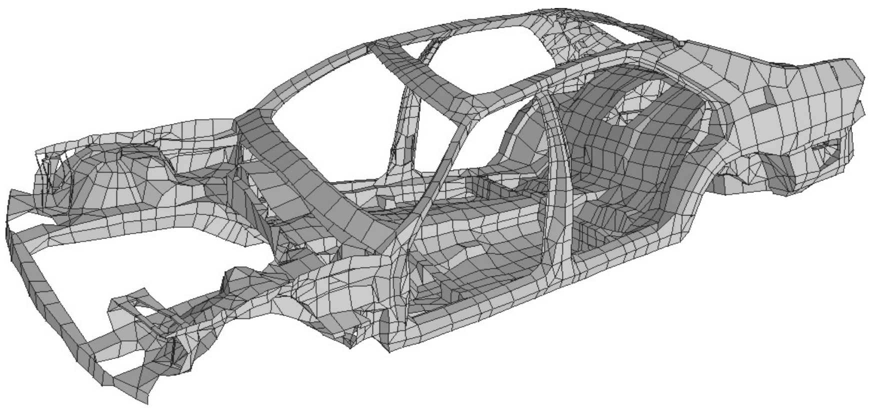
Disclaimer: this tutorial is not intending to give a complete overview of all publications on the subject, but to introduce the main concepts in substructuring …

1. **Reduction of dynamic models**
2. **Substructuring – a random walk in history**
3. **Assembly of substructures**
4. **Reduction of substructure dynamics**
5. **Experimental substructuring**

1. Reduction of Dynamic Models

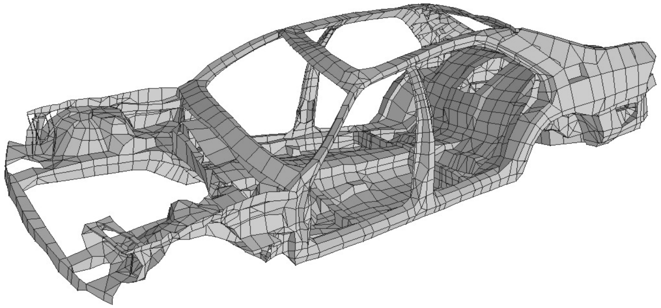
Models very refined
→ to represent complex geometry
→ to compute static stresses

But much **too refined** for representing **global dynamics**!

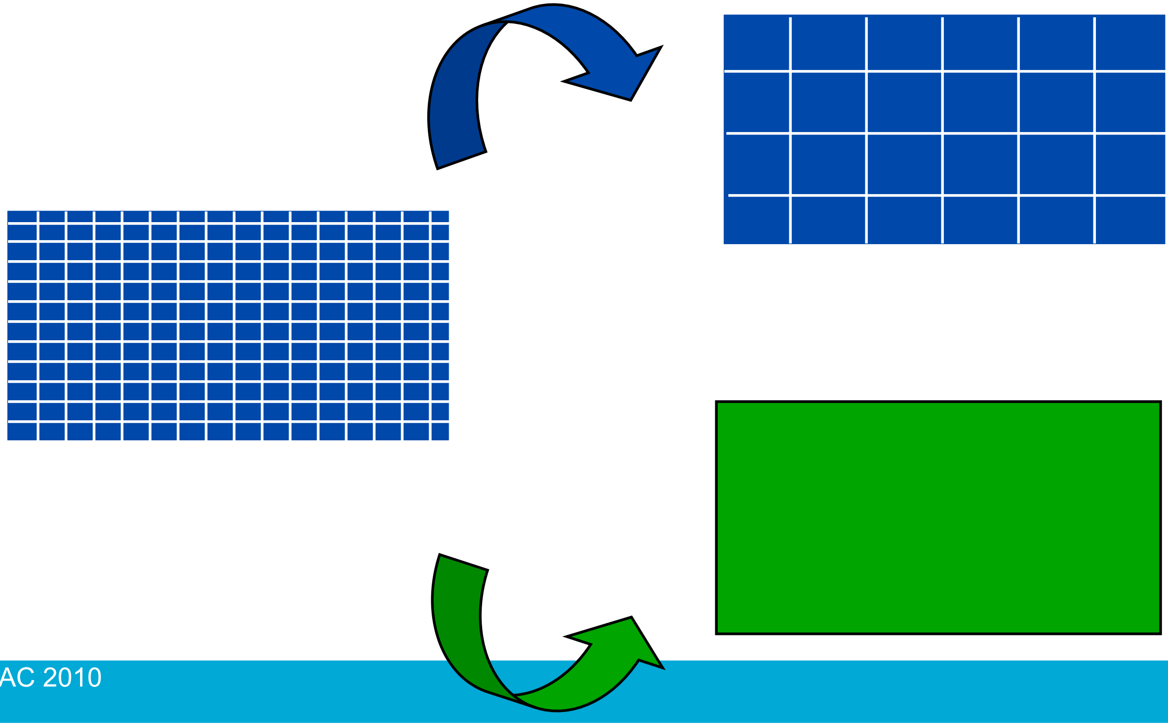


CU, Boulder

1. Reduction of Dynamic Models



2 options to have a lower order model:



New mesh

Reducing the mathematical space of dofs

$$q_{fine} = R\, q_{coarse}$$

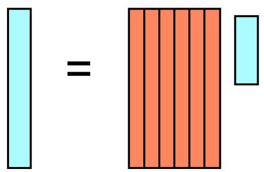
1. Reduction of Dynamic Models

Assume a large linear(ized) dynamic model:

$$M\ddot{q} + D\dot{q} + Kq = f$$

Try the approximation

$$q = R\hat{q}$$



"Ritz vectors"

Force error due to approximation

$$MR\ddot{\hat{q}} + DR\dot{\hat{q}} + KR\hat{q} = f + r$$

Require that error is 0 in a subspace

$$\tilde{R}^T r = 0$$

$$\tilde{R}^T (MR\ddot{\hat{q}} + DR\dot{\hat{q}} + KR\hat{q}) = \tilde{R}^T f$$

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1. Reduction of Dynamic Models

$$M\ddot{q} + D\dot{q} + Kq = f$$

$$\Big\downarrow \quad q = R\hat{q}$$

$$\boxed{\hat{M}\ddot{\hat{q}} + \hat{D}\dot{\hat{q}} + \hat{K}\hat{q} = \hat{f}}$$

where

$$\hat{M} = \tilde{R}^T M R$$

$$\hat{D} = \tilde{R}^T D R$$

$$\hat{K} = \tilde{R}^T K R$$

$$\hat{f} = \tilde{R}^T f$$

- Small system (say 200 dofs)
- $\hat{q}$ also called reduced or generalized dofs (*q* called the physical dofs)
- Remember that there is always a related force error involved (residue)
- For symmetrical systems, one usually chooses $\tilde{R} = R$
- Once $\hat{q}$ is computed, the "physical dofs" are found by substitution in $q = R\hat{q}$

1. Reduction of Dynamic Models

$$M\ddot{q} + D\dot{q} + Kq = f$$

$$q = R\hat{q}$$

$$\boxed{\hat{M}\ddot{\hat{q}} + \hat{D}\dot{\hat{q}} + \hat{K}\hat{q} = \hat{f}}$$

Which reduction basis $\boldsymbol{R}$?

✓ $\hat{M}, \hat{D}, \hat{K}$ sparse

✓ must well approximate the solution

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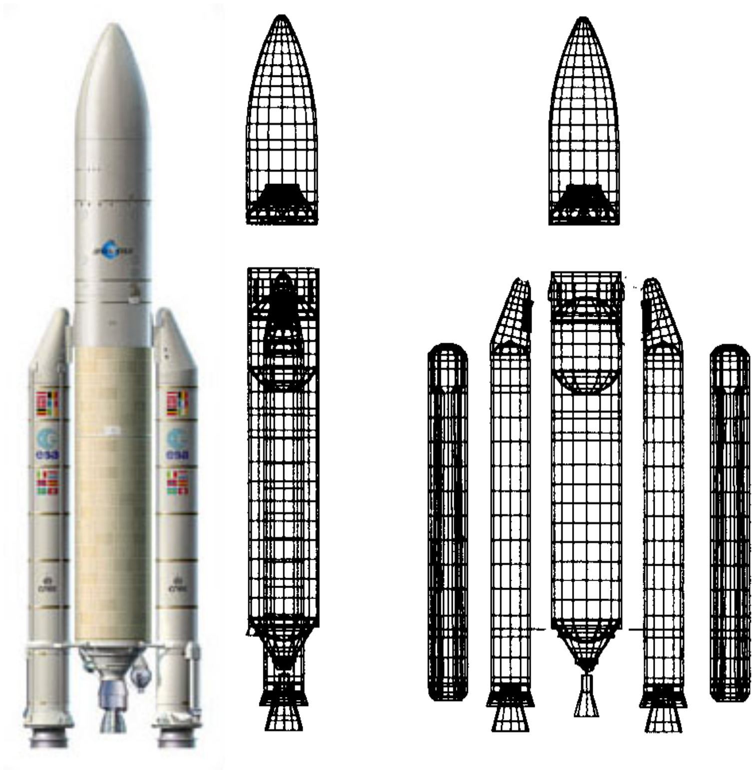
1. Reduction of Dynamic Models

Idea : *cut in pieces* and *reduce every sub-part*



Dynamic substructuring

- Natural approach if model already decomposed
- Allows isolating difficult regions (contact, NL …)
- Suitable for parallel computing
- Easy re-use of components
- Allows combining numerical/experimental submodels

General concepts in substructuring

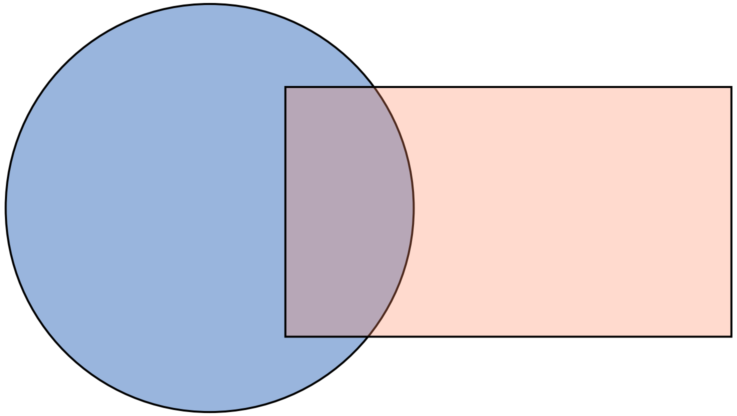
1. Reduction of dynamic models
2. Substructuring – a random walk in history
3. Assembly of substructures
4. Numerical reduction
5. Experimental substructuring

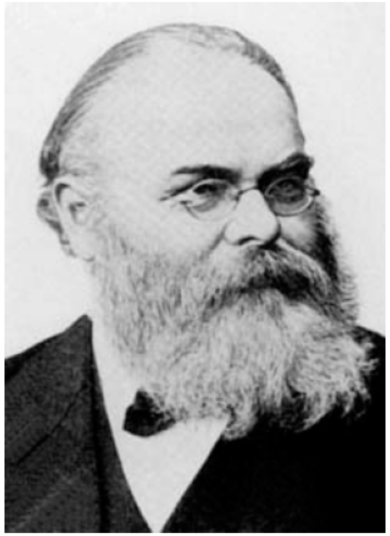
2. Substructuring – A random walk in history

The paradigm of ***“Divide and Conquer”***

❑ Already promoted by Julius Caesar himself …

❑ But the first mathematical application of it was by H. Schwarz, in 1890, who wanted to prove the existence and uniqueness of the solution to the Laplace equation in a complex domain:





2. Substructuring – A random walk in history

- Later, with the work of Courant (1943), the idea of subdivision was used to define Ritz approximation per elements.
Finite Elements were born …

- In the 60's the idea of partitioning a finite element model in substructures was proposed to reduce the complexity of the models

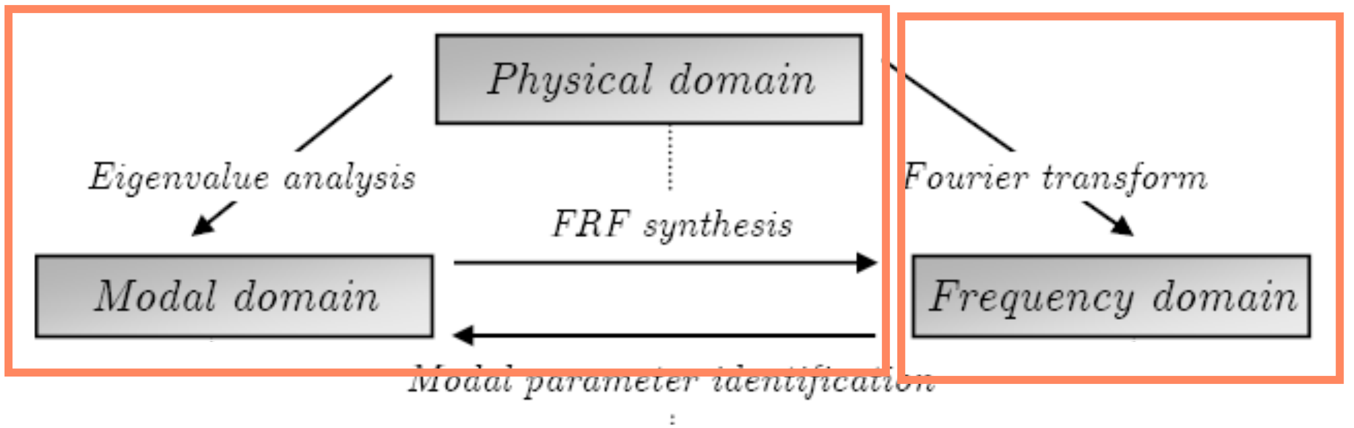
- In the 80's, in order to speed-up the solution of linear algebraic system, like in $Ku=F$, the computational domain was cut in sub-domains in order to share work amongst several CPUs.
That was the start of *Domain Decomposition.*

- End of the 60's, but especially in the 80's, decomposing a structure in experimental mechanics was applied in order to simplify the testing

2. Substructuring – General framework

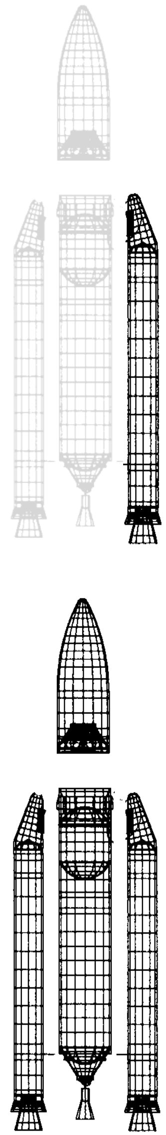
Reduction of Numerical models

Experimental models



Representation of substructure
(reduced or not)

Assembly of Substructures



General concepts in substructuring

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5. **Experimental substructuring**

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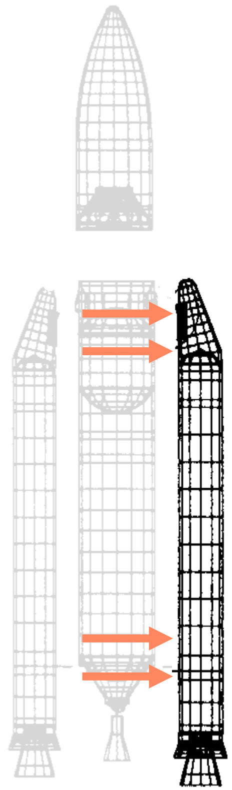
3. Assembly

$$M^{(s)}\ddot{u}^{(s)}(t) + C\dot{u}(t) + K^{(s)}u^{(s)}(t) = f^{(s)}(t) + g^{(s)}(t)$$

Interface (internal) forces



Interface forces

- enforce the compatibility:

$$u^{(k)} - u^{(l)} = 0$$

- are in equilibrium (action-reaction)

$$g^{(k)} + g^{(l)} = 0.$$

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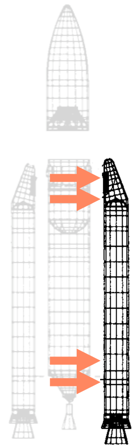
3. Assembly

$$M^{(s)}\ddot{u}^{(s)}(t) + C\dot{u}(t) + K^{(s)}u^{(s)}(t) = f^{(s)}(t) + g^{(s)}(t)$$

$$u^{(k)} - u^{(l)} = 0$$

$$g^{(k)} + g^{(l)} = 0.$$



In block diagonal matrix form this can be written as

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = 0 \\ L^T g = 0 \end{cases}$$

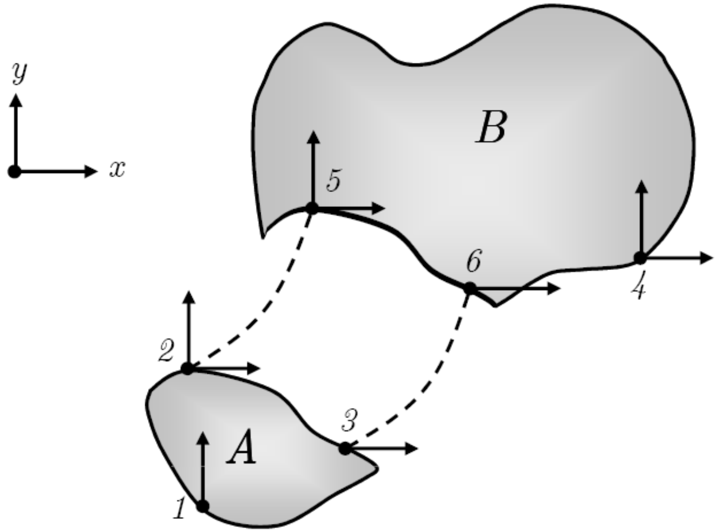
Signed Boolean matrix

Boolean matrix

where

$$M \triangleq \begin{bmatrix} M^{(1)} & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & M^{(n)} \end{bmatrix} \quad u \triangleq \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(n)} \end{bmatrix}$$

3. Assembly



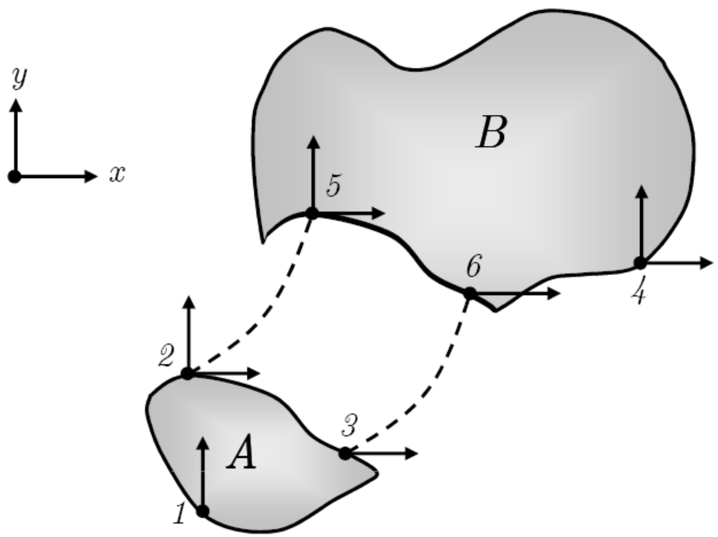
Example

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = 0 \\ L^T g = 0 \end{cases}$$

Signed Boolean matrix

$$B = \begin{array}{c} \begin{array}{ccccccccc} u_{1y} & u_{2x} & u_{2y} & u_{3x} & u_{4x} & u_{4y} & u_{5x} & u_{5y} & u_{6x} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

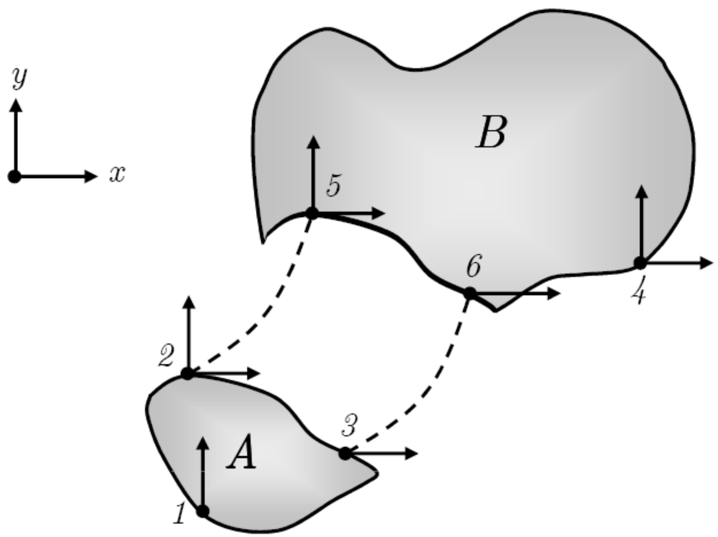
3. Assembly



Example

$$
\boldsymbol{L}^T \boldsymbol{g} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g_{2x} \\ g_{2y} \\ g_{3x} \\ 0 \\ 0 \\ g_{5x} \\ g_{5y} \\ g_{6x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_{2x} + g_{5x} \\ g_{2y} + g_{5y} \\ g_{3x} + g_{6x} \end{bmatrix} = \boldsymbol{0}
$$

3. Assembly



Example

L is clearly the assembly matrix known from F.E.

Hence one also verifies that

$$\boldsymbol{u} = \boldsymbol{L}\boldsymbol{q} = \begin{bmatrix} u_{1y} \\ u_{5x} = u_{2x} \\ u_{5y} = u_{2y} \\ u_{6x} = u_{3x} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \\ u_{6x} \end{bmatrix}$$

Unique dofs
for total structure

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{B}\boldsymbol{L}\boldsymbol{q} = \boldsymbol{0} \qquad \forall \boldsymbol{q}$$

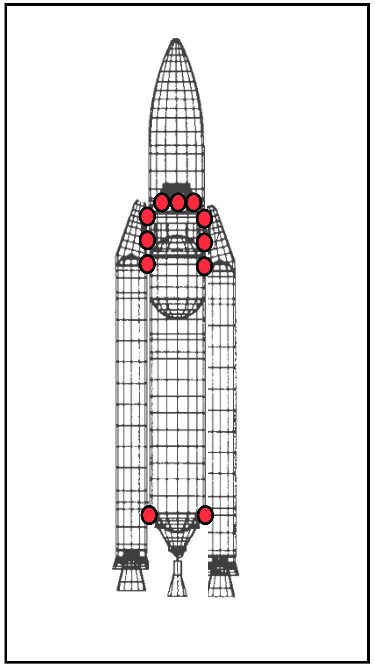
L is the NULL SPACE of ***B***

3. Assembly

Recall matrix notation (called 3-field formulation)

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = 0 \\ L^T g = 0 \end{cases}$$

If one assumes that there is a unique set of dofs on the interface: $u = Lq$



$$\begin{cases} L^T\left(ML\ddot{q} + CL\dot{q} + KLq = f + g\right) \\ L^T g = 0 \end{cases}$$

$$\boxed{\tilde{M}\ddot{q} + \tilde{C}\dot{q} + \tilde{K}q = \tilde{f}}$$

where $\tilde{M} \triangleq L^T M L \quad \ldots$

Primal assembly (see F.E.)

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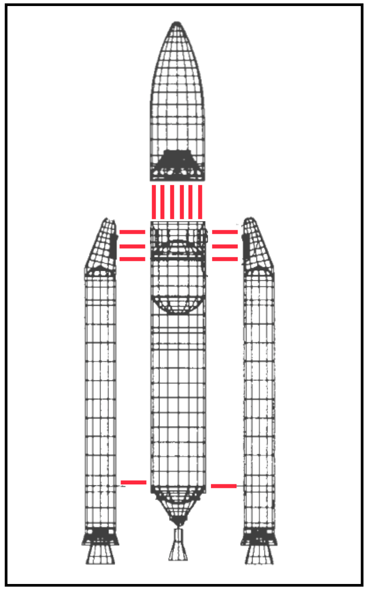
3. Assembly

Recall back the block 3-field formulation

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = 0 \\ L^T g = 0 \end{cases}$$

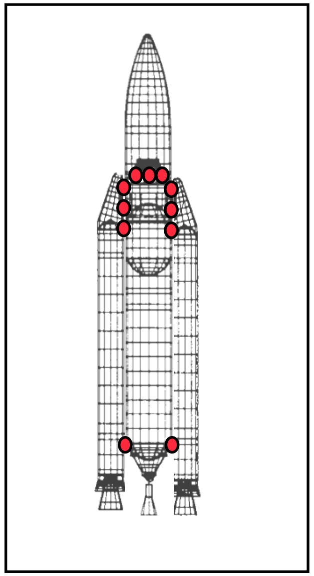
If one assumes that there the interface forces are in equilibrium: $g = -B^T\lambda$



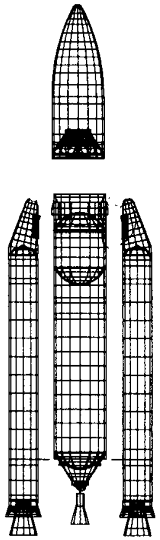
$$\boxed{\begin{cases} M\ddot{u} + C\dot{u} + Ku + B^T\lambda = f \\ Bu = 0 \end{cases}}$$

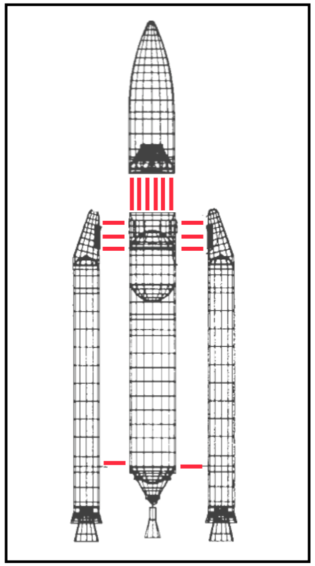
DUAL assembly

3. General framework – Coupling of physical matrices
Summary



$$
\begin{cases}
M\ddot{u} + C\dot{u} + Ku = f + g \\
Bu = 0 \quad \text{Interface compatibility} \\
L^T g = 0 \quad \text{Interface equilibrium}
\end{cases}
$$





3-field formulation

a priori compatible

a priori in equil.

$$\tilde{M}\ddot{q} + \tilde{C}\dot{q} + \tilde{K}q = \tilde{f}$$

Primal assembly

$$
\begin{cases}
M\ddot{u} + C\dot{u} + Ku + B^T\lambda = f \\
Bu = 0
\end{cases}
$$

DUAL assembly

End-result obviously the same
Dual assembly works also if non-matching meshes!

$$
\begin{bmatrix}
K^{(1)} & & & \\
 & \ddots & & \\
 & & & K^{(N_s)}
\end{bmatrix}
$$

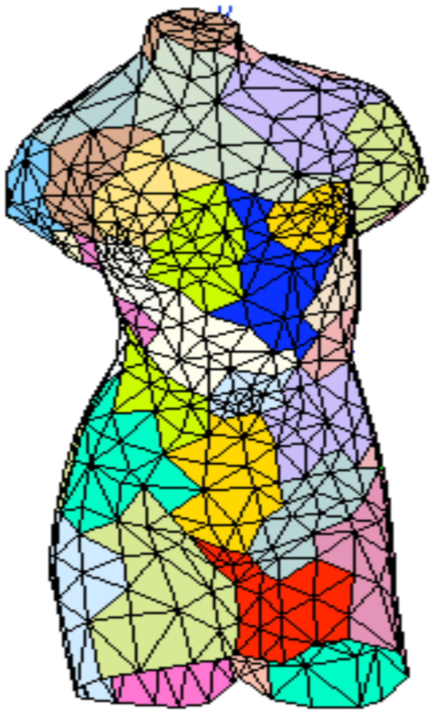
$$
\begin{bmatrix}
K^{(1)} & & & \mathbf{0} \\
 & \ddots & & \mathbf{I} \\
 & & & \mathbf{0} \\
 & & K^{(N_s)} & \mathbf{-I} \\
\mathbf{0} \quad \mathbf{I} & \mathbf{0} & \mathbf{-I} & \mathbf{0}
\end{bmatrix}
$$

More details? …………….. see then

A General Framework for Dynamic Substructuring
-A history, review and classification of DS techniques –

D. de Klerk, D.J. Rixen and S.N. Voormeeren

Time for a 10 seconds break …



… you are doing great, just breath deeply …

General concepts in substructuring

1. **Reduction of dynamic models**

2. **Substructuring – a random walk in history**

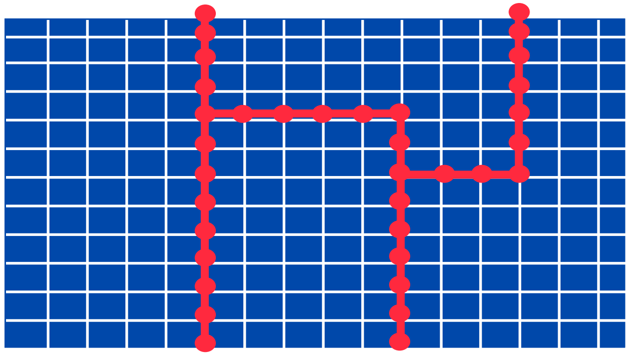
3. **Assembly of substructures**

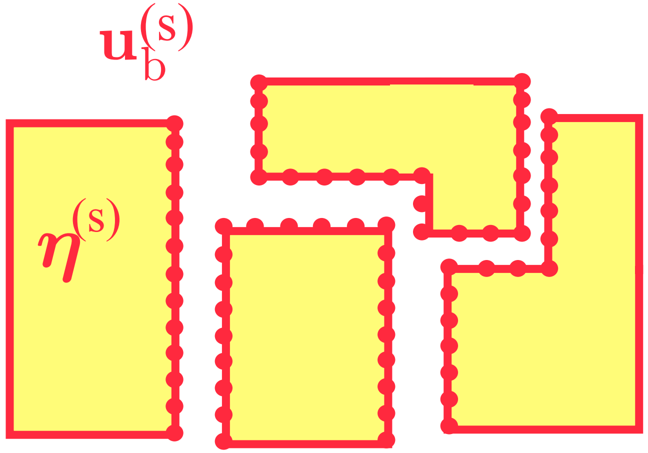
4. **Reduction of substructure dynamics**
   - ***The Craig- Bampton CMS***
   - ***Variants of the Craig-Bampton method***
   - ***Other ingredients for the CMS: free interface modes***
   - ***The issue of interface reduction***

5. **Experimental substructuring**

Super-elements

Can be assembled like normal FE:
Super (or macro) elements





$$\boldsymbol{u} \simeq \boldsymbol{R}\eta$$

$$\begin{bmatrix} \boldsymbol{u}^{(1)} \\ \boldsymbol{u}^{(2)} \\ \vdots \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^{(1)} & & & \\ & \boldsymbol{R}^{(2)} & & \\ & & \boldsymbol{R}^{(3)} & \\ & & & \vdots \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_b^{(1)} \\ \boldsymbol{\eta}^{(1)} \\ \boldsymbol{u}_b^{(2)} \\ \boldsymbol{\eta}^{(2)} \\ \vdots \end{bmatrix}$$

An example of super-element: the Craig-Bampton CMS

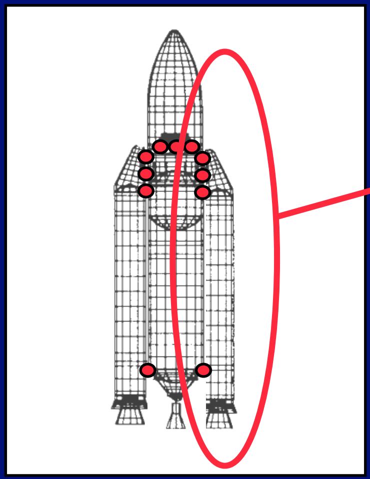
Roy Craig is one of the fathers of dynamic substructuring.
See also his book and review papers.

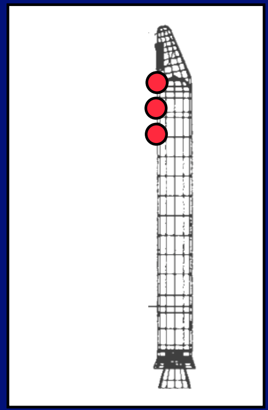
This Component Modes Synthesis was published in 1968 and developed while Craig was a young engineer at Boeing.

The Craig-Bampton method is one of the most used techniques because it yields “nice” reduced matrices for the super-element.

Many of the CMS methods are variants of the Craig-Bampton method

Craig-Bampton component mode synthesis (CMS)





Boundary displacements $\boldsymbol{q}_b^{(s)}$

Internal displacements $\boldsymbol{q}_i^{(s)}$

In each substructure *(s)* :

$$\boldsymbol{M}_{ii}^{(s)}\ddot{\boldsymbol{q}}_i^{(s)} + \boldsymbol{K}_{ii}^{(s)}\boldsymbol{q}_i^{(s)} = -\boldsymbol{K}_{ib}^{(s)}\boldsymbol{q}_b^{(s)} - \boldsymbol{M}_{ib}^{(s)}\ddot{\boldsymbol{q}}_b^{(s)}$$

$\boldsymbol{q}_i^{(s)}$ computed by modal superposition of ***fixed interface modes*** :

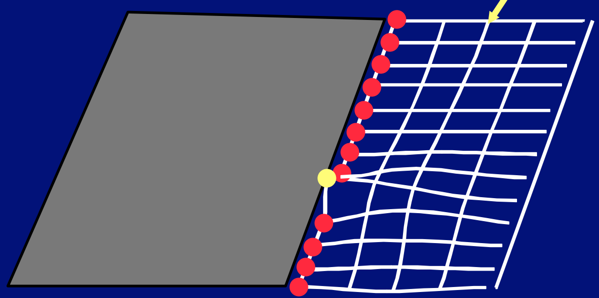
$$\boldsymbol{q}_i^{(s)} = -\boldsymbol{K}_{ii}^{(s)^{-1}}\boldsymbol{K}_{ib}^{(s)}\,\boldsymbol{q}_b^{(s)} + \boldsymbol{\Phi}^{(s)}\boldsymbol{\eta}^{(s)}$$

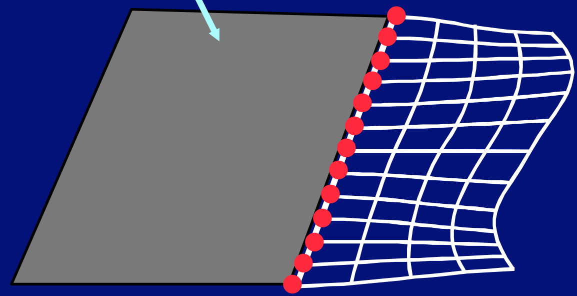
[Craig-Bampton]

static solution

Craig-Bampton component mode synthesis (CMS)

$$
\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_b \\ \boldsymbol{q}_i^{(1)} \\ \vdots \\ \boldsymbol{q}_i^{(N_s)} \end{bmatrix} \simeq \underbrace{\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{\Psi}^{(1)} & \boldsymbol{\Phi}^{(1)} & & \boldsymbol{0} \\ \vdots & & \ddots & \\ \boldsymbol{\Psi}^{(N_s)} & \boldsymbol{0} & & \boldsymbol{\Phi}^{(N_s)} \end{bmatrix}}_{\boldsymbol{R}_{CB}} \begin{bmatrix} \boldsymbol{q}_b \\ \boldsymbol{\eta}^{(1)} \\ \vdots \\ \boldsymbol{\eta}^{(N_s)} \end{bmatrix}
$$





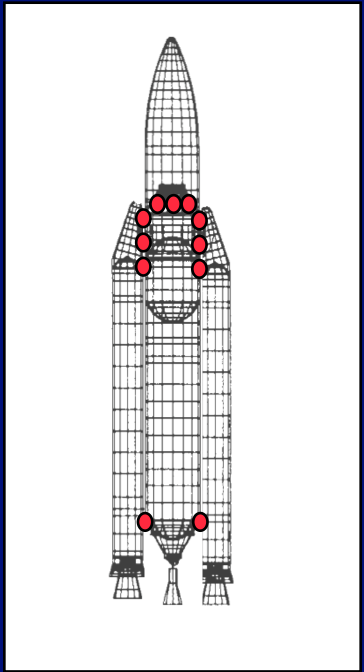
$$\boldsymbol{\Psi}^{(s)} = -\boldsymbol{K}_{ii}^{(s)^{-1}} \boldsymbol{K}_{ib}^{(s)}$$

Static modes
(all)

$$\left(\boldsymbol{K}_{ii}^{(s)} - \omega_k^{(s)^2} \boldsymbol{M}_{ii}^{(s)}\right) \phi_k^{(s)} = \boldsymbol{0}$$

Vibration modes (fixed interface)
(*k<N*)

Craig-Bampton component mode synthesis (CMS)



$$\bar{\boldsymbol{K}}_{CB} = \boldsymbol{R}_{CB}^{T}\boldsymbol{K}\,\boldsymbol{R}_{CB} = \begin{bmatrix} \boldsymbol{S}_{bb} & & & \boldsymbol{0} \\ & \boldsymbol{\Omega}^{(1)^2} & & \\ & & \ddots & \\ \boldsymbol{0} & & & \boldsymbol{\Omega}^{(N_s)^2} \end{bmatrix}$$

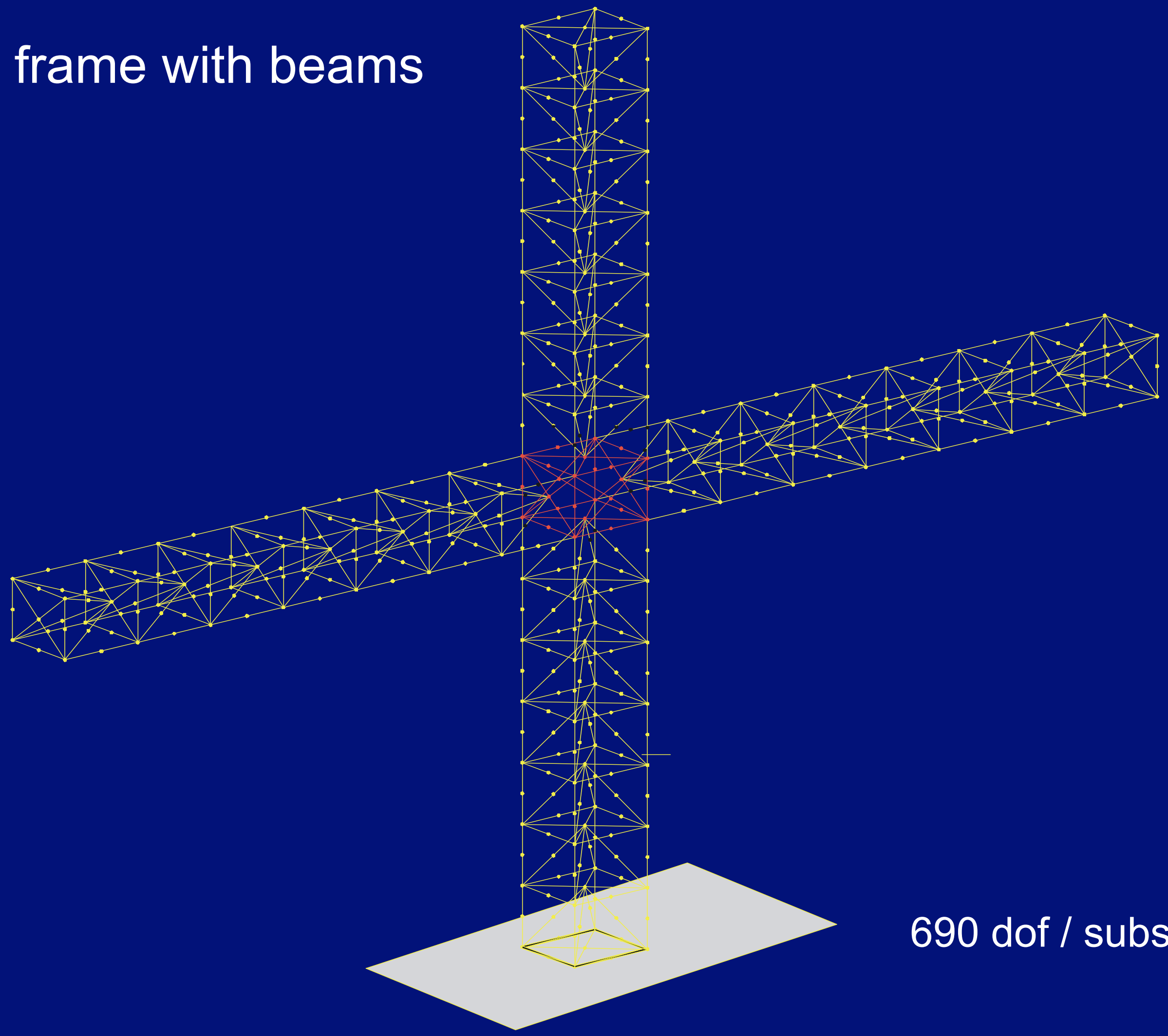
$$\bar{\boldsymbol{M}}_{CB} = \boldsymbol{R}_{CB}^{T}\boldsymbol{M}\,\boldsymbol{R}_{CB} = \begin{bmatrix} \boldsymbol{M}_{bb}^{*} & \boldsymbol{M}_{b\phi}^{(1)} & \dots & \boldsymbol{M}_{b\phi}^{(N_s)} \\ \boldsymbol{M}_{\phi b}^{(1)} & \boldsymbol{I} & & \boldsymbol{0} \\ \vdots & & \ddots & \\ \boldsymbol{M}_{\phi b}^{(N_s)} & \boldsymbol{0} & & \boldsymbol{I} \end{bmatrix}$$

$\boldsymbol{S}_{bb}$ $\boldsymbol{M}_{bb}^{*}$: assembly of statically condensed matrices (Schur complement, Guyan)

- K reduces to a quasi-diagonal matrix
- Mass coupling between boundary and internal d.o.f.
- Good approximation with a small number of modes
- Building the reduced system is not very expensive

An example of super-element: the Craig-Bampton CMS

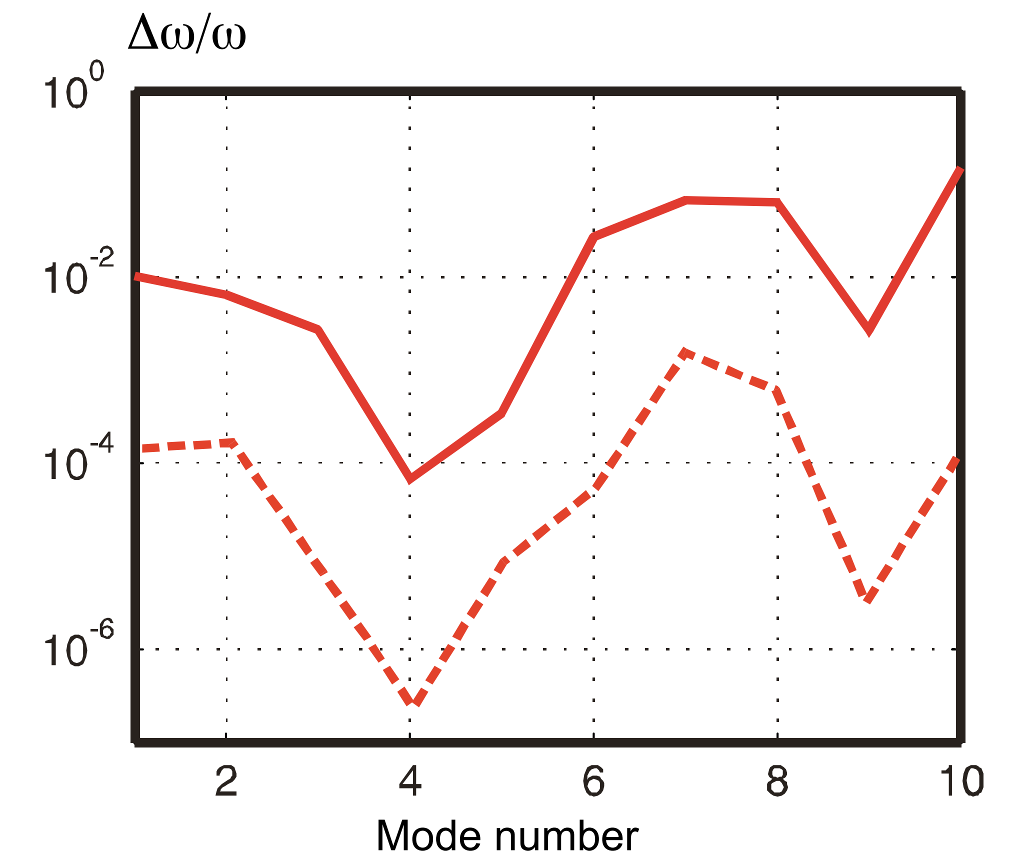
Truss frame with beams

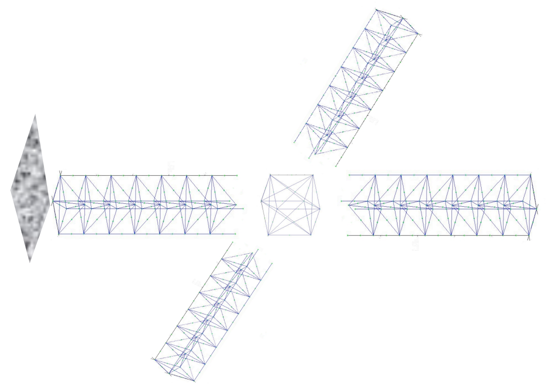


690 dof / substructure

An example of super-element: the Craig-Bampton CMS

Relative error on frequencies





Craig-Bampton

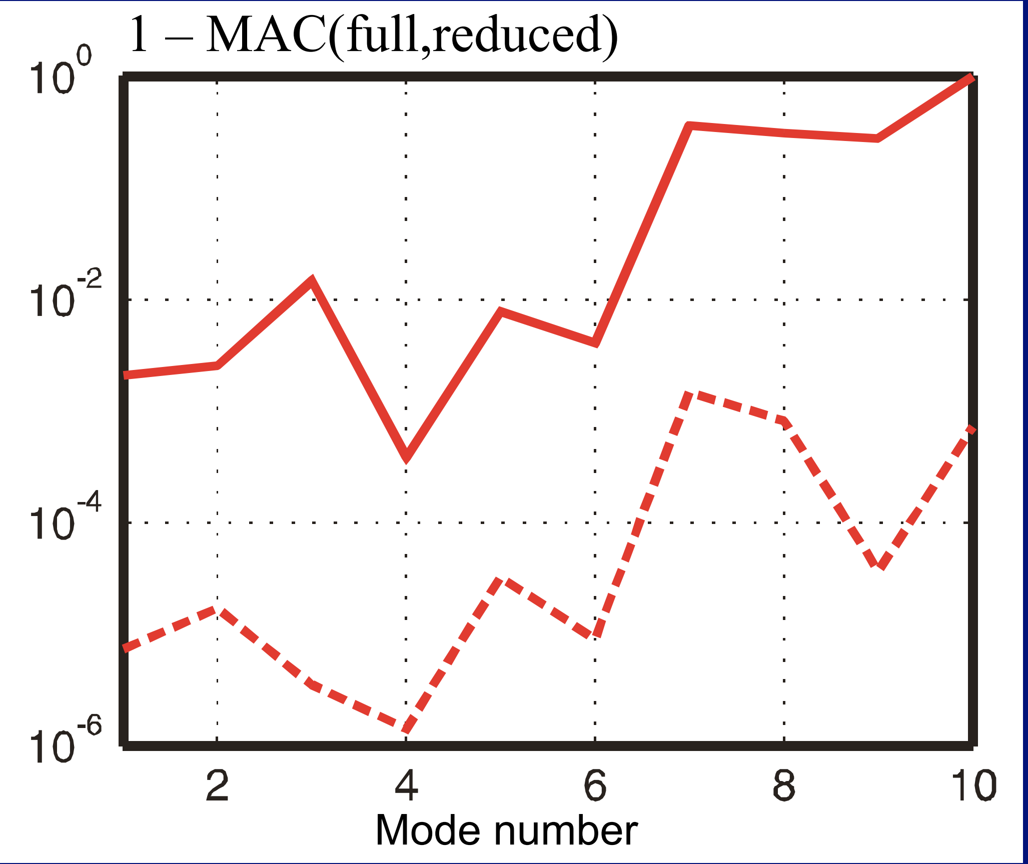
4 modes / sub-structure

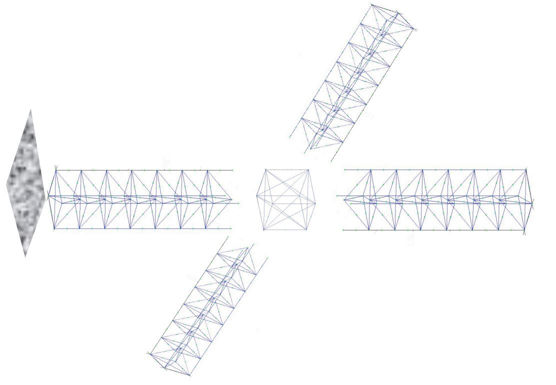
Craig-Bampton

10 modes / sub-structure

An example of super-element: the Craig-Bampton CMS

Modes error





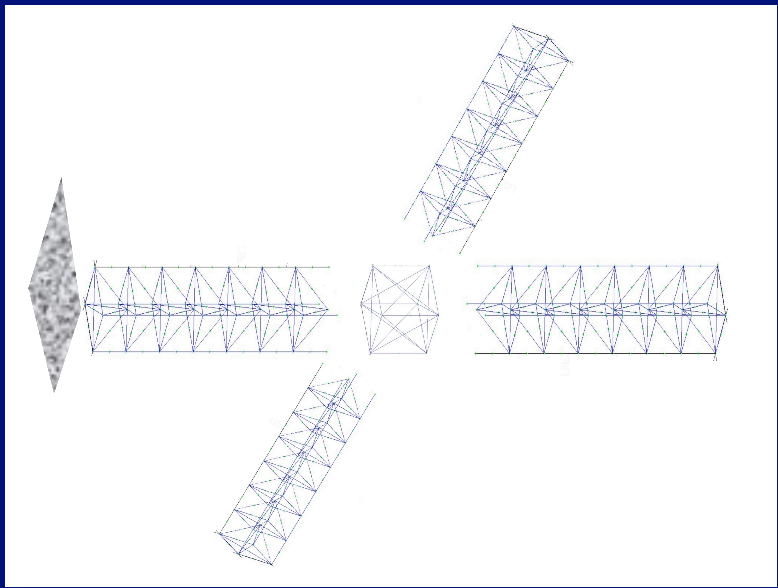
Craig-Bampton

4 modes / sub-structure

Craig-Bampton

10 modes / sub-structure

An example of super-element: the Craig-Bampton CMS



How to choose the substructure modes to be included in the reduction ?

- Rule of thumb: include substructure modes having a frequency less than
  1.8 X global eigenfrequency to ebe computed

- Evaluate how much the internal modes participate to the representation of the substructure mass as seen from the interface (effective modal mass)

- Evaluate how much reaction forces an internal mode generates on the interface

- *A posteriori,* it is always possible to check the residual force due to the approximation

General concepts in substructuring

1. Reduction of dynamic models

2. Substructuring – a random walk in history

3. Assembly of substructures

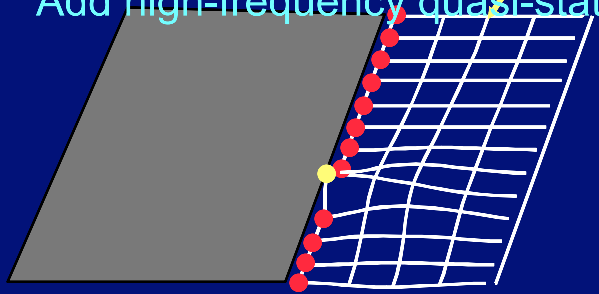
4. Reduction of substructure dynamics
    - *The Craig- Bampton CMS*
    - *Variants of the Craig-Bampton method*
    - *Other ingredients for the CMS: free interface modes*
    - *The issue of interface reduction*

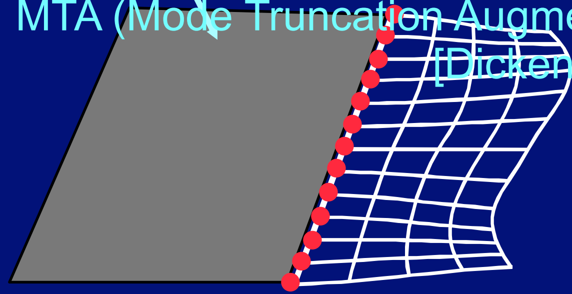
5. Experimental substructuring

Variants on the Craig-Bampton CMS

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_b \\ \boldsymbol{q}_i^{(1)} \\ \vdots \\ \boldsymbol{q}_i^{(N_s)} \end{bmatrix} \simeq \underbrace{\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{\Psi}^{(1)} & \boldsymbol{\Phi}^{(1)} & & \boldsymbol{0} \\ \vdots & & \ddots & \\ \boldsymbol{\Psi}^{(N_s)} & \boldsymbol{0} & & \boldsymbol{\Phi}^{(N_s)} \end{bmatrix}}_{\boldsymbol{T}_{CB}} \begin{bmatrix} \boldsymbol{q}_b \\ \boldsymbol{\eta}^{(1)} \\ \vdots \\ \boldsymbol{\eta}^{(N_s)} \end{bmatrix}$$

Add high-frequency quasi-static corrections: MTA (Mode Truncation Augmentation)
[Dickens,Rixen]





Variants on the Craig-Bampton CMS

$$q = \begin{bmatrix} q_b \\ q_i^{(1)} \\ \vdots \\ q_i^{(N_s)} \end{bmatrix} = \begin{bmatrix} I \\ \Psi^{(1)} \\ \vdots \\ \Psi^{(N_s)} \end{bmatrix} q_b$$

$$\cancel{q \simeq \begin{bmatrix} I & 0 & \cdots & 0 \\ \Psi^{(1)} & \Phi^{(1)} & & 0 \\ \vdots & & \ddots & \\ \Psi^{(N_s)} & 0 & & \Phi^{(N_s)} \end{bmatrix} \begin{bmatrix} q_b \\ \eta^{(1)} \\ \vdots \\ \eta^{(N_s)} \end{bmatrix}}$$

$$q \simeq \begin{bmatrix} I & 0 & \cdots & 0 \\ \Psi^{(1)} & X^{(1)} & & \\ \vdots & & \ddots & \\ \Psi^{(N_s)} & 0 & & X^{(N_s)} \end{bmatrix} \begin{bmatrix} q_b \\ \eta^{(1)} \\ \vdots \\ \eta^{(N_s)} \end{bmatrix}$$

Start with a pure Guyan reduction

Compute approximate dynamics of full system

Substitute in original full problem

$$M^{(s)} R\ddot{\eta}^{(s)} + C R\dot{\eta} + K^{(s)} R\eta^{(s)} = f^{(s)} + g^{(s)} + r^{(s)}$$

Compute residual forces per substructure

Compute the static response of
the susbtructures to that force error

$X^{(s)}$

Enrich the reduction basis

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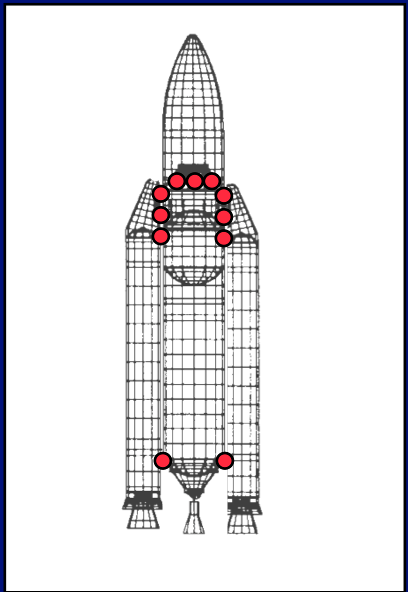
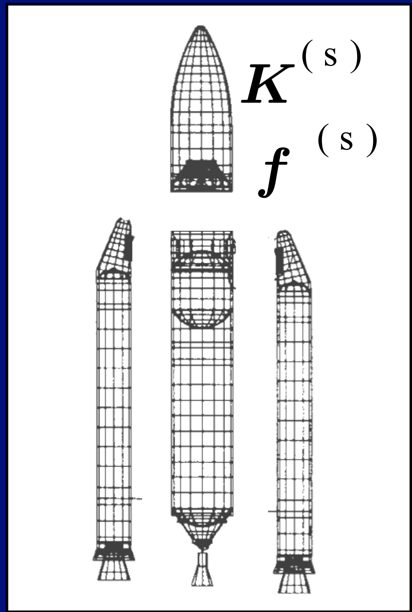
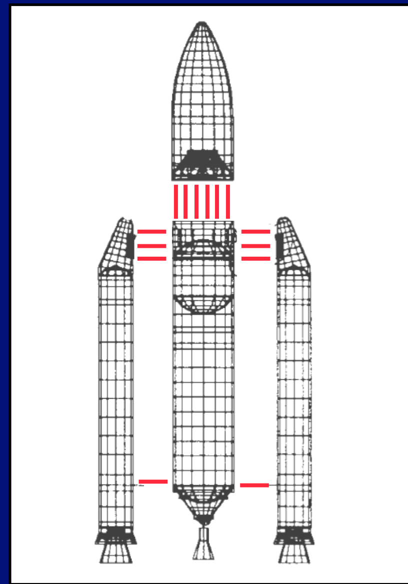
   - *The issue of interface reduction*

5. Experimental substructuring

Other ingredients for CMS: free interface modes







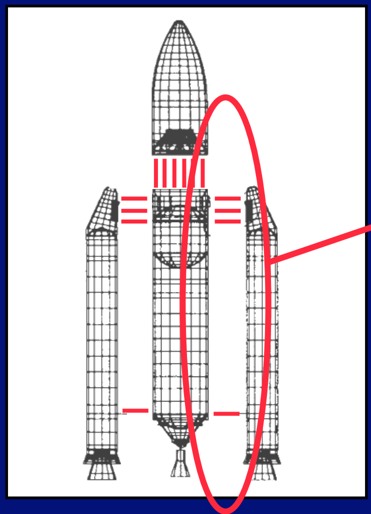
Primal

Dual

$$K_g = \begin{bmatrix} K^{(1)} & & & \\ & \ddots & & \\ & & & K^{(N_s)} \end{bmatrix}$$

$$\begin{bmatrix} K^{(1)} & & & \begin{matrix}0\\ I\end{matrix} \\ & \ddots & & \\ & & K^{(N_s)} & \begin{matrix}0\\ -I\end{matrix} \\ 0 \quad I & & 0 \quad -I & 0 \end{bmatrix} \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N_s)} \\ \lambda \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N_s)} \\ 0 \end{bmatrix}$$

4. Other ingredients for CMS: free interface modes



$$\left\{\begin{array}{l} \boldsymbol{M}^{(s)}\ddot{\boldsymbol{u}}^{(s)} + \boldsymbol{K}^{(s)}\boldsymbol{u}^{(s)} = -\begin{bmatrix} \boldsymbol{B}^{(s)^T}\boldsymbol{\lambda} \\ \boldsymbol{0} \end{bmatrix} \\ \displaystyle\sum_{s=1}^{N_s} \boldsymbol{B}^{(s)}\boldsymbol{u}^{(s)} = \boldsymbol{0} \end{array}\right.$$

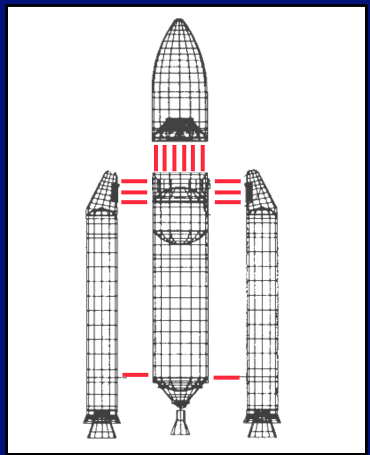
Boolean

$\boldsymbol{u}^{(s)}$ computed by mode superposition of ***free interface modes*** :

Generalized inverse

$$\boldsymbol{u}^{(s)} = \underbrace{-\boldsymbol{K}^{(s)^+}\boldsymbol{B}^{(s)^T}\boldsymbol{\lambda} + \boldsymbol{R}^{(s)}\boldsymbol{\alpha}^{(s)}}_{\boldsymbol{u}_{stat}^{(s)}} + \boldsymbol{\Theta}^{(s)}\boldsymbol{\eta}^{(s)}$$

4. Other ingredients for CMS: free interface modes

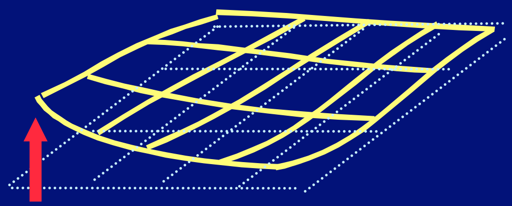


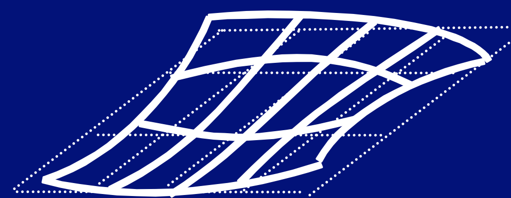
Generalized inverse (Factorization + fictitious links)

$$\boldsymbol{u}^{(s)} = \underbrace{-\boldsymbol{K}^{(s)^{+}} \boldsymbol{B}^{(s)^{T}}} \boldsymbol{\lambda} + \boldsymbol{R}^{(s)} \boldsymbol{\alpha}^{(s)} + \boldsymbol{\Theta}^{(s)} \boldsymbol{\eta}^{(s)}$$

Interface flexibility modes

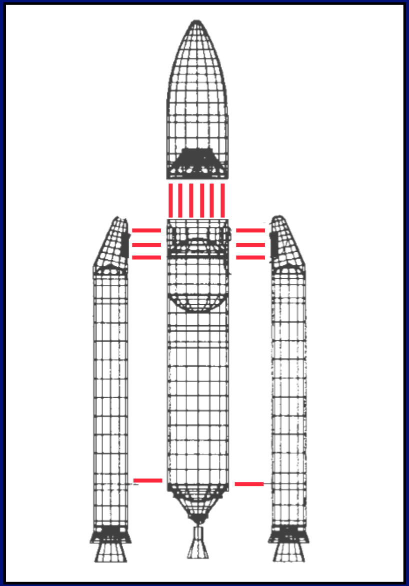
Free interface vibration modes





$$\left(\boldsymbol{K}^{(s)} - \omega_r^{(s)^2} \boldsymbol{M}^{(s)}\right) \theta_r^{(s)} = \boldsymbol{0}$$

4. Other ingredients for CMS: free interface modes



$$\boldsymbol{u}^{(s)} = -\boldsymbol{K}^{(s)^{+}} \boldsymbol{B}^{(s)^{T}} \boldsymbol{\lambda} + \boldsymbol{R}^{(s)} \boldsymbol{\alpha}^{(s)} + \boldsymbol{\Theta}^{(s)} \boldsymbol{\eta}^{(s)}$$

$\longleftrightarrow$

$$\boldsymbol{u}^{(s)} = -G_{res}^{(s)} \boldsymbol{B}^{(s)^{T}} \boldsymbol{\lambda} + \boldsymbol{R}^{(s)} \boldsymbol{\alpha}^{(s)} + \boldsymbol{\Theta}^{(s)} \boldsymbol{\eta}^{(s)}$$

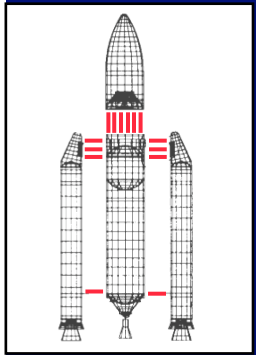
$$\boldsymbol{K}^{(s)^{+}} - \sum_{r=1}^{q} \frac{\boldsymbol{\theta}_r^{(s)^{T}} \boldsymbol{\theta}_r^{(s)}}{\omega_r^{(s)^{2}}}$$

Residual flexibility modes
(attachment modes)

such that

$$\begin{aligned} \boldsymbol{\Theta}^{(s)T} \boldsymbol{K}^{(s)} \boldsymbol{G}_{res}^{(s)} &= \boldsymbol{0} \\ \boldsymbol{\Theta}^{(s)T} \boldsymbol{M}^{(s)} \boldsymbol{G}_{res}^{(s)} &= \boldsymbol{0} \\ \boldsymbol{R}^{(s)T} \boldsymbol{M}^{(s)} \boldsymbol{G}_{res}^{(s)} &= \boldsymbol{0} \end{aligned}$$

4. Other ingredients for CMS: free interface modes



$$u^{(s)} = T^{(s)} \begin{bmatrix} \alpha^{(s)} \\ \eta^{(s)} \\ \lambda \end{bmatrix} = \begin{bmatrix} R^{(s)} & \Theta^{(s)} & G_{res}^{(s)} B^{(s)^T} \end{bmatrix} \begin{bmatrix} \alpha^{(s)} \\ \eta^{(s)} \\ \lambda \end{bmatrix}$$

Mac-Neal, Rubin, Craig-Chang :

“ *eliminate l in term of displacements dof per substructure* “

→ Obtain a super-element

Dual Craig-Bampton :

Use dual assembly and reduce the problem

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku + B^T\lambda = f \\ Bu = 0 \end{cases}$$

General concepts in substructuring

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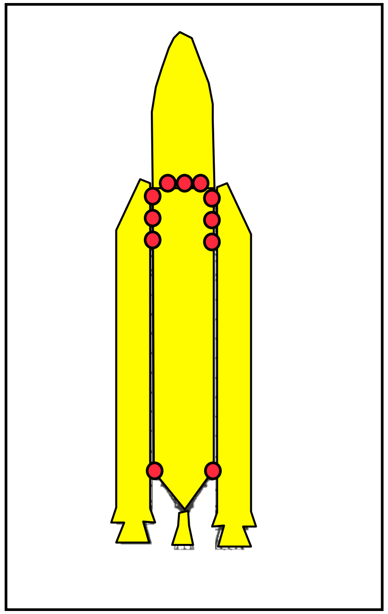
5. Experimental substructuring

The issue of interface reduction



Limitation:

If number of substructures increase,

number of $\boldsymbol{u}_b$ increases !

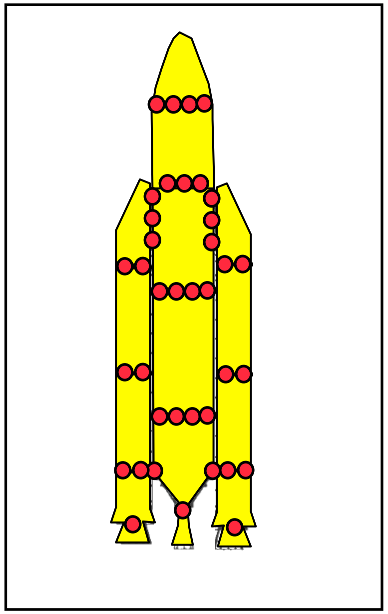
Interface reduction :

Define interface modes :

$$\boldsymbol{u}_b \simeq \boldsymbol{\Phi}_b \boldsymbol{\eta}_b$$

$$\bar{\bar{\boldsymbol{M}}}_{CB} = \begin{bmatrix} \boldsymbol{\Phi}_b^T & \boldsymbol{I} \end{bmatrix} \bar{\boldsymbol{M}}_{CB} \begin{bmatrix} \boldsymbol{\Phi}_b \\ \boldsymbol{I} \end{bmatrix}$$

$$\bar{\bar{\boldsymbol{K}}}_{CB} = \begin{bmatrix} \boldsymbol{\Phi}_b^T & \boldsymbol{I} \end{bmatrix} \bar{\boldsymbol{K}}_{CB} \begin{bmatrix} \boldsymbol{\Phi}_b \\ \boldsymbol{I} \end{bmatrix}$$



General concepts in substructuring

1. Reduction of dynamic models

2. Substructuring – a random walk in history

3. Assembly of substructures

4. Reduction of substructure dynamics

5. Experimental substructuring
   - A guitar
   - A car

Frequency Based Substructuring

$$\begin{cases} \boldsymbol{Z}(\omega)\boldsymbol{u}(\omega) = \boldsymbol{f}(\omega) + \boldsymbol{B}^{\boldsymbol{T}}\lambda \\ \boldsymbol{B}\boldsymbol{u}(\omega) = \boldsymbol{0} \end{cases}$$

Measured FRFs

$$\begin{cases} \boldsymbol{u}(\omega) = \boldsymbol{Y}(\omega)\{\boldsymbol{f} + \boldsymbol{B}^{\boldsymbol{T}}\lambda\} \\ \boldsymbol{B}\boldsymbol{u}(\omega) = \boldsymbol{0} \end{cases}$$

$$\boxed{\boldsymbol{u} = \boldsymbol{Y}\boldsymbol{f} - \boldsymbol{Y}\boldsymbol{B}^{T}(\boldsymbol{B}\boldsymbol{Y}\boldsymbol{B}^{T})^{-1}\boldsymbol{B}\boldsymbol{Y}\boldsymbol{f}}$$

Guitar test case

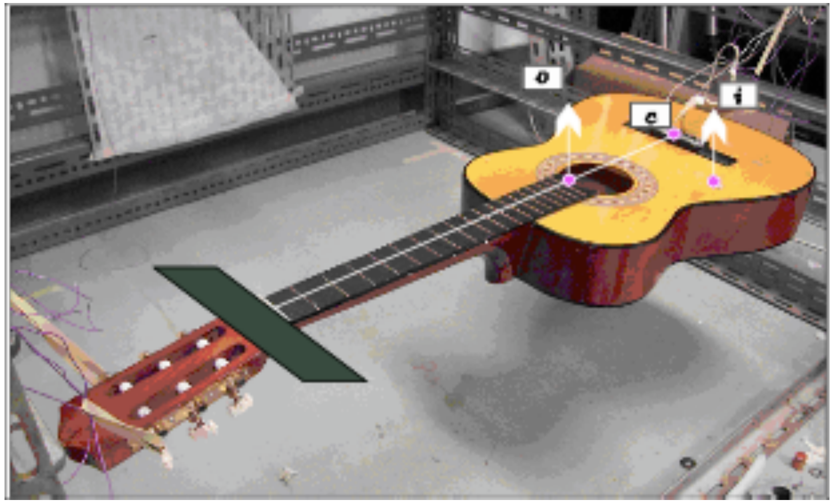
Dynamic combination of guitar body and strings determines the sound quality

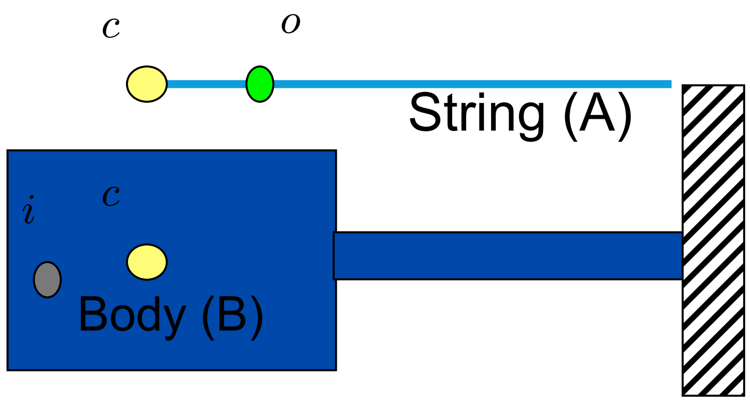
TU Delft
Delft University of Technology

Guitar test case: substructure model

Input on sound table / output on cord (easier to measure)

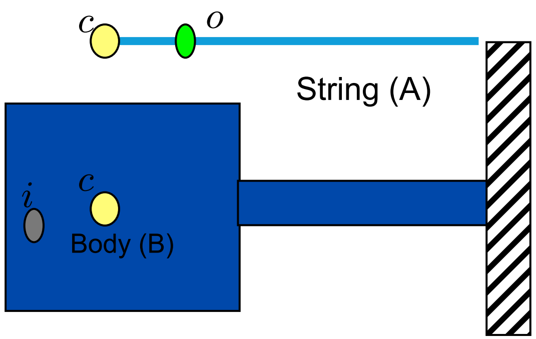
- Assume string and nut clamped (negligible dynamic contribution)
- Only one string present
- In-plane vibrations are assumed negligible
- Analytical model for string / experimental model for body
- Impedance head and laser vibrometer (to minimize additional mass effects)

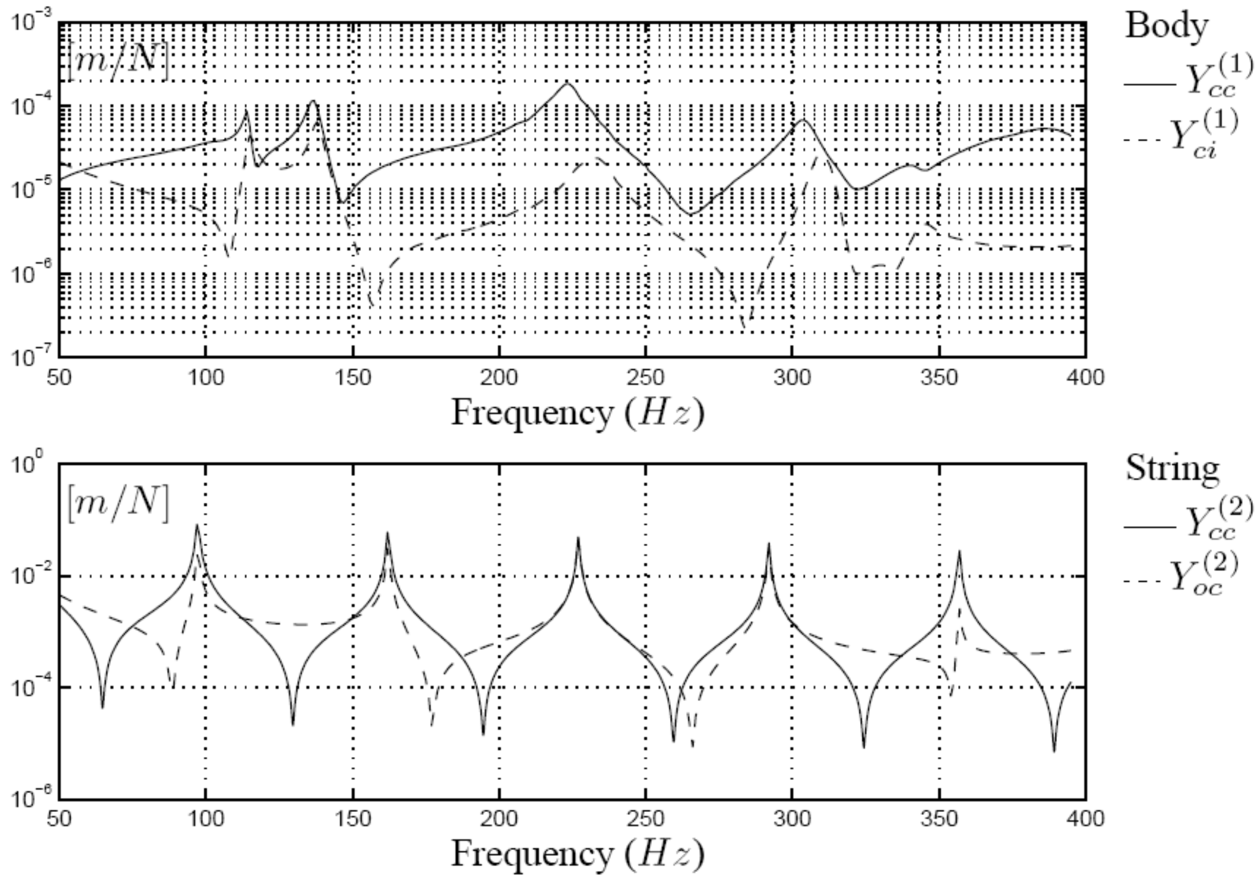




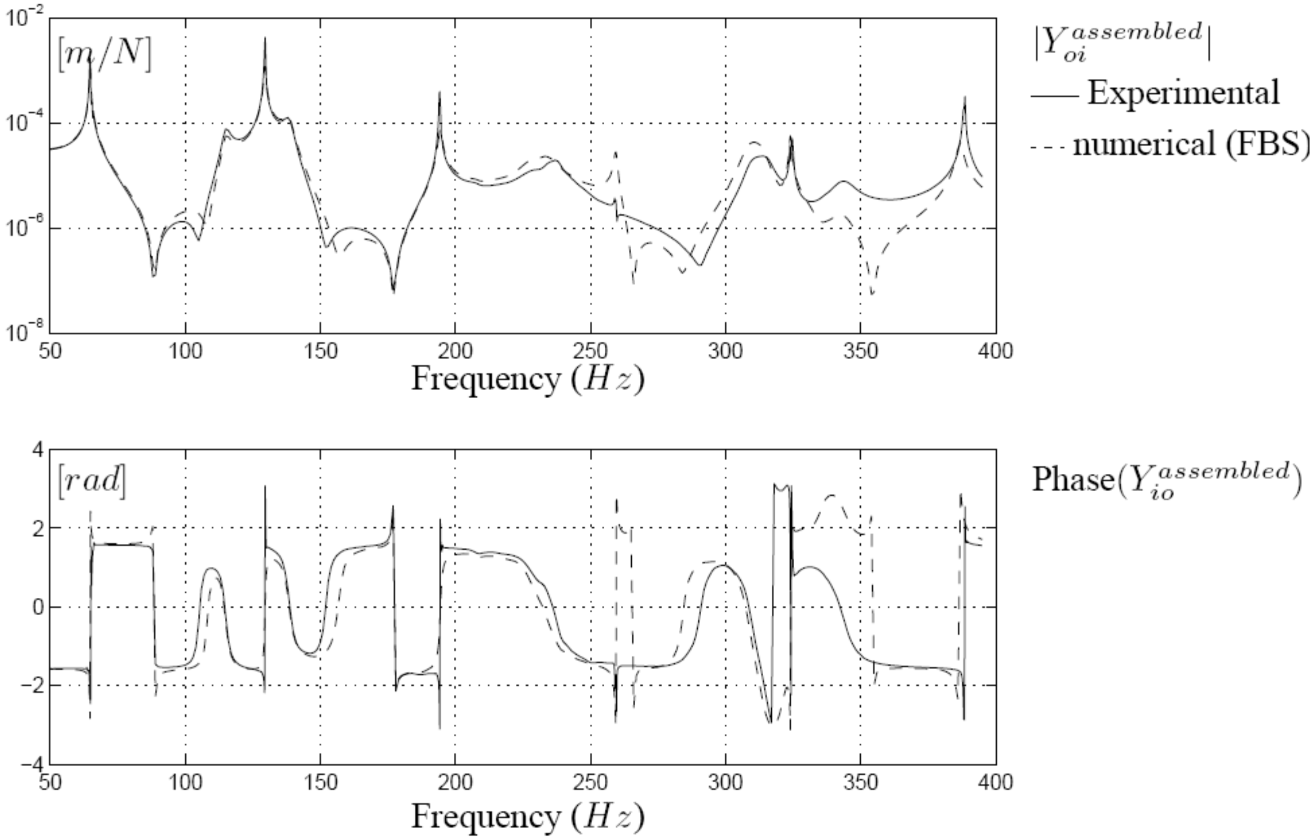
Guitar test case: FRF guitar body

Measured by admittance head and laser





Guitar test case: assembled

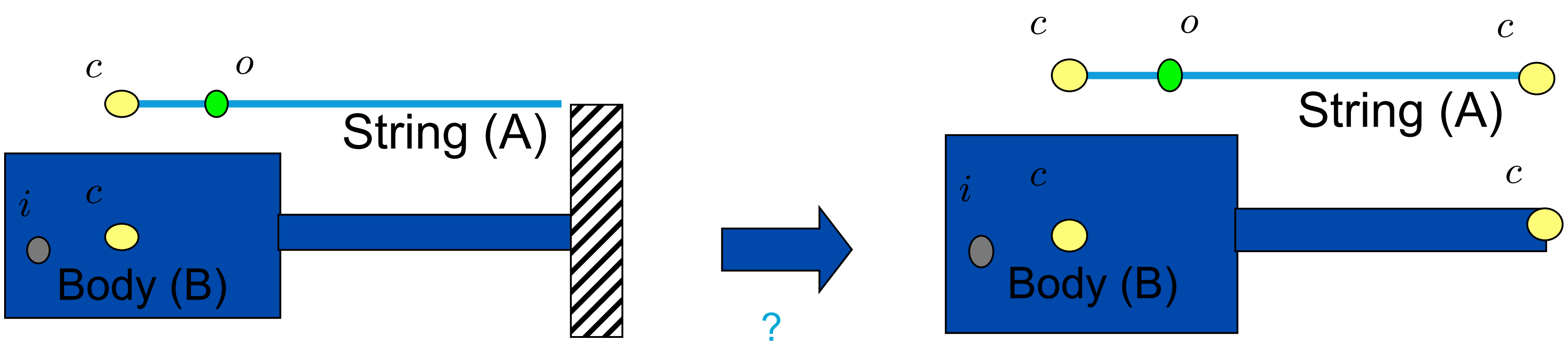


Good up to 250 Hz. Bad above. Why ?

Guitar test case

Good up to 250 Hz. Bad above. Why ?

- ✓ Prestress of sound table ?
- ✓ Vibro-acoustic coupling ?
- ✓ Out-of-plane motion ?
- ✓ Validity of substructure model ?



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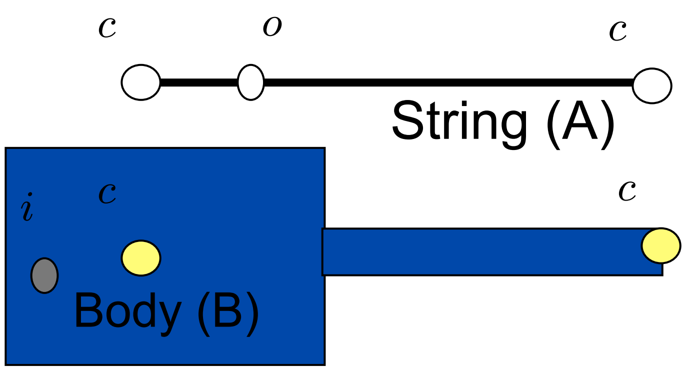
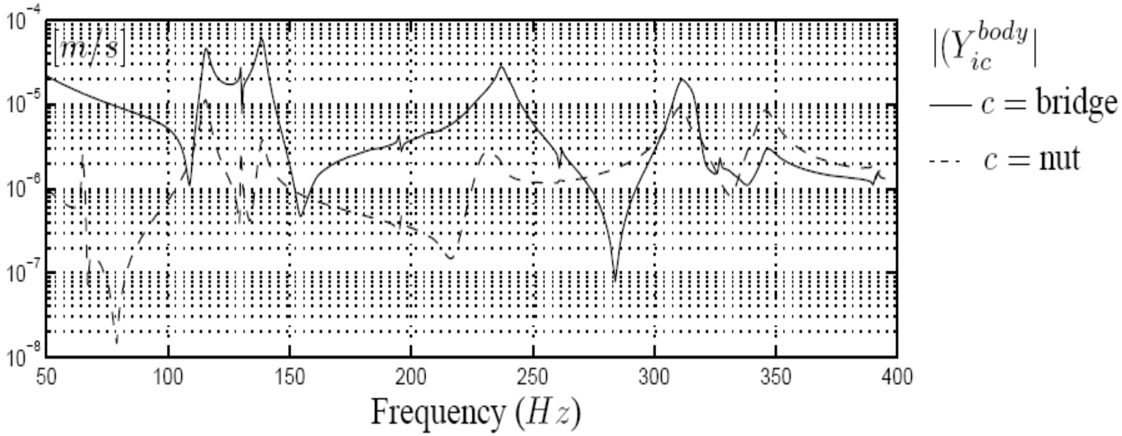
Guitar test case

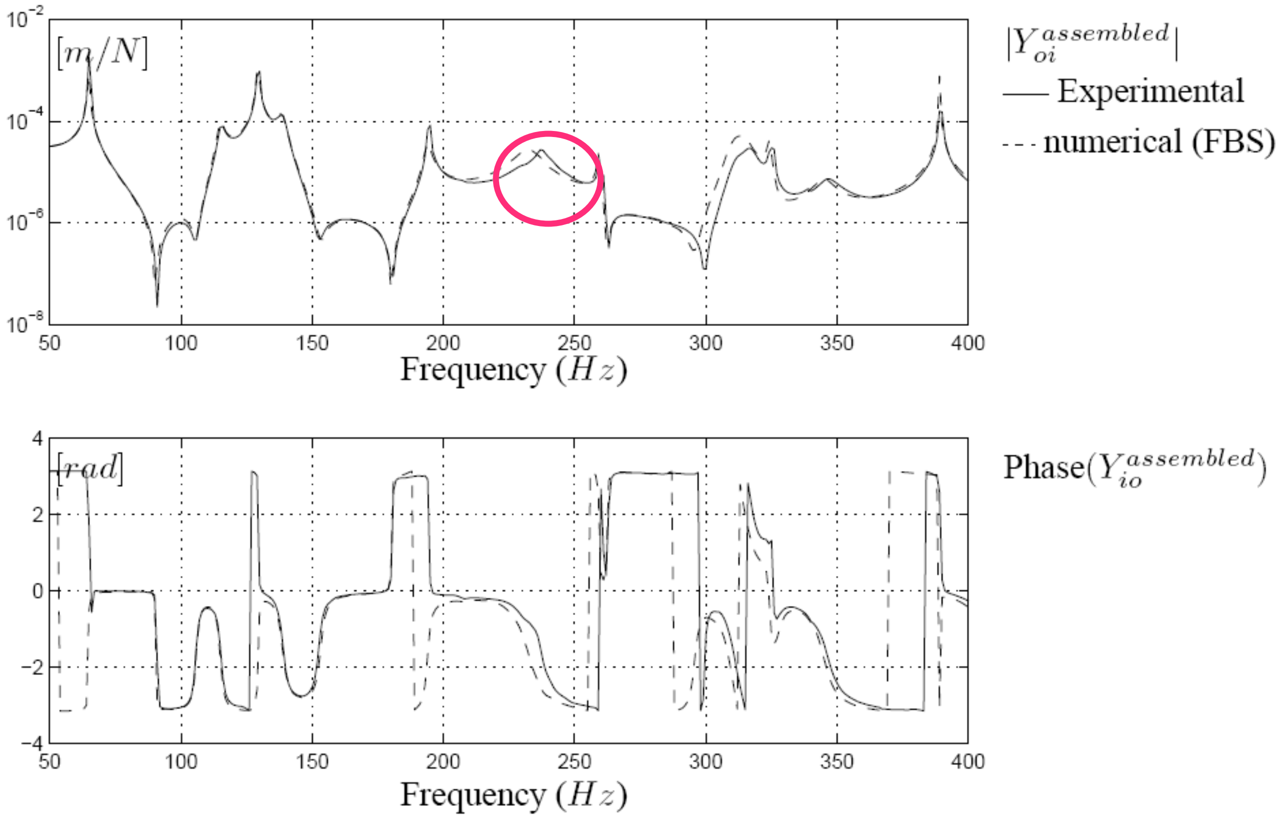




Important dynamics at nut above 250 Hz !

Redo analysis with 2 connection points

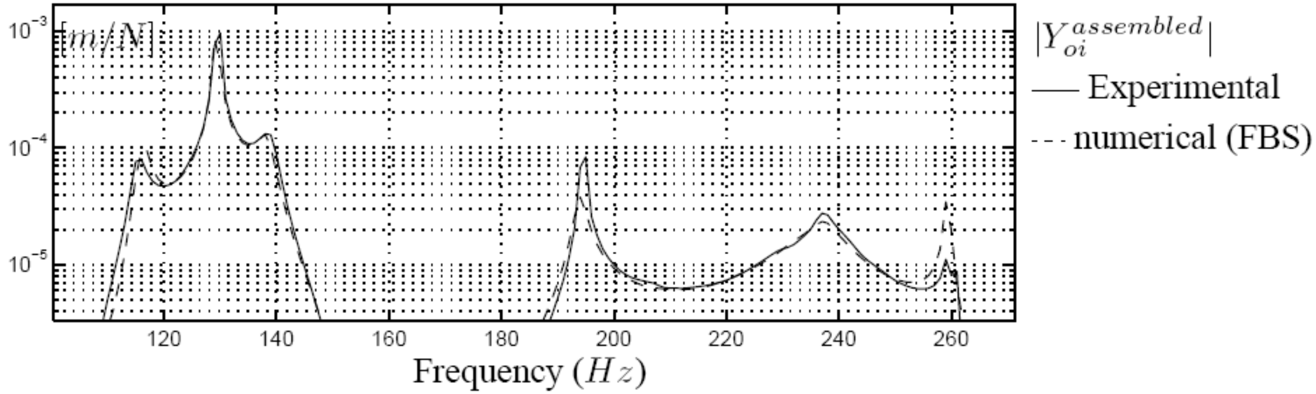
Guitar test case



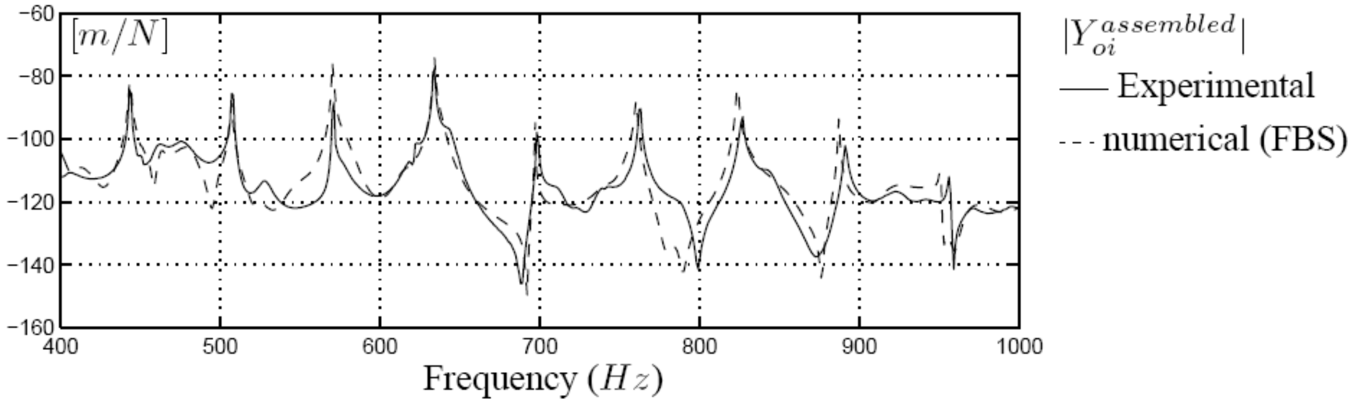
Very good agreement … except around 230 Hz → Loose bridge saddle !

Guitar test case

After re-gluing the bridge saddle:



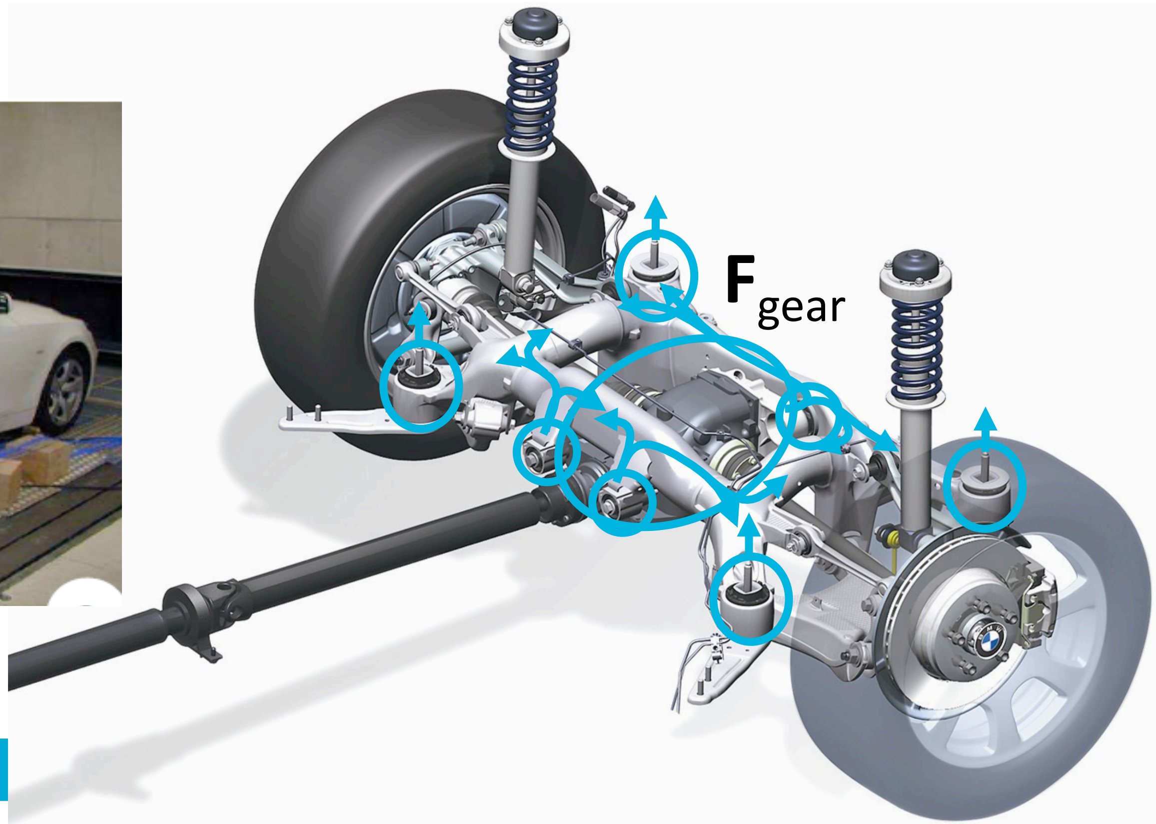
Even at high frequency !



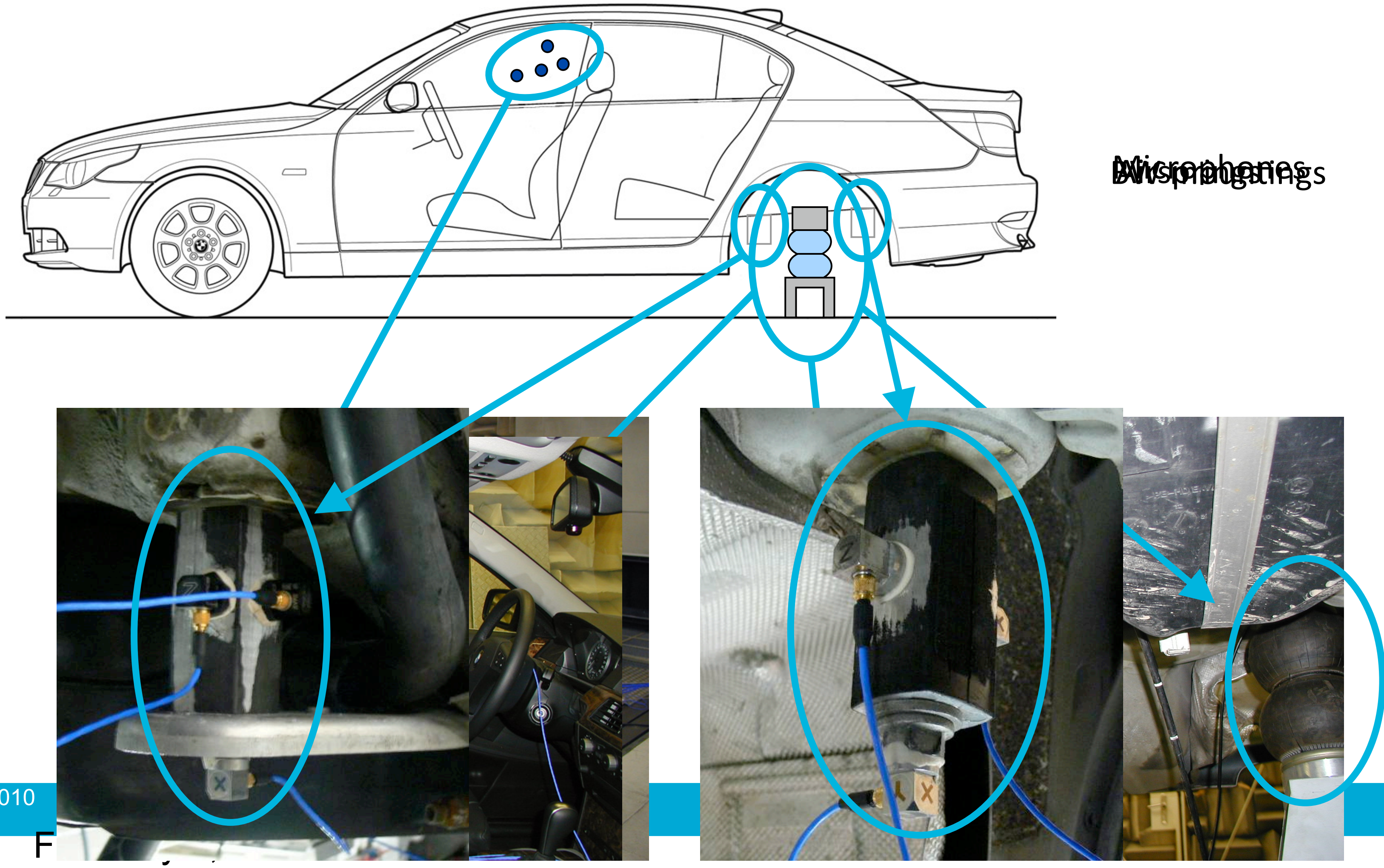
Modeling of Substructures
Gear noise propagation path – Description

Rear-axle differential (RAD)

mounting (RAD-M)



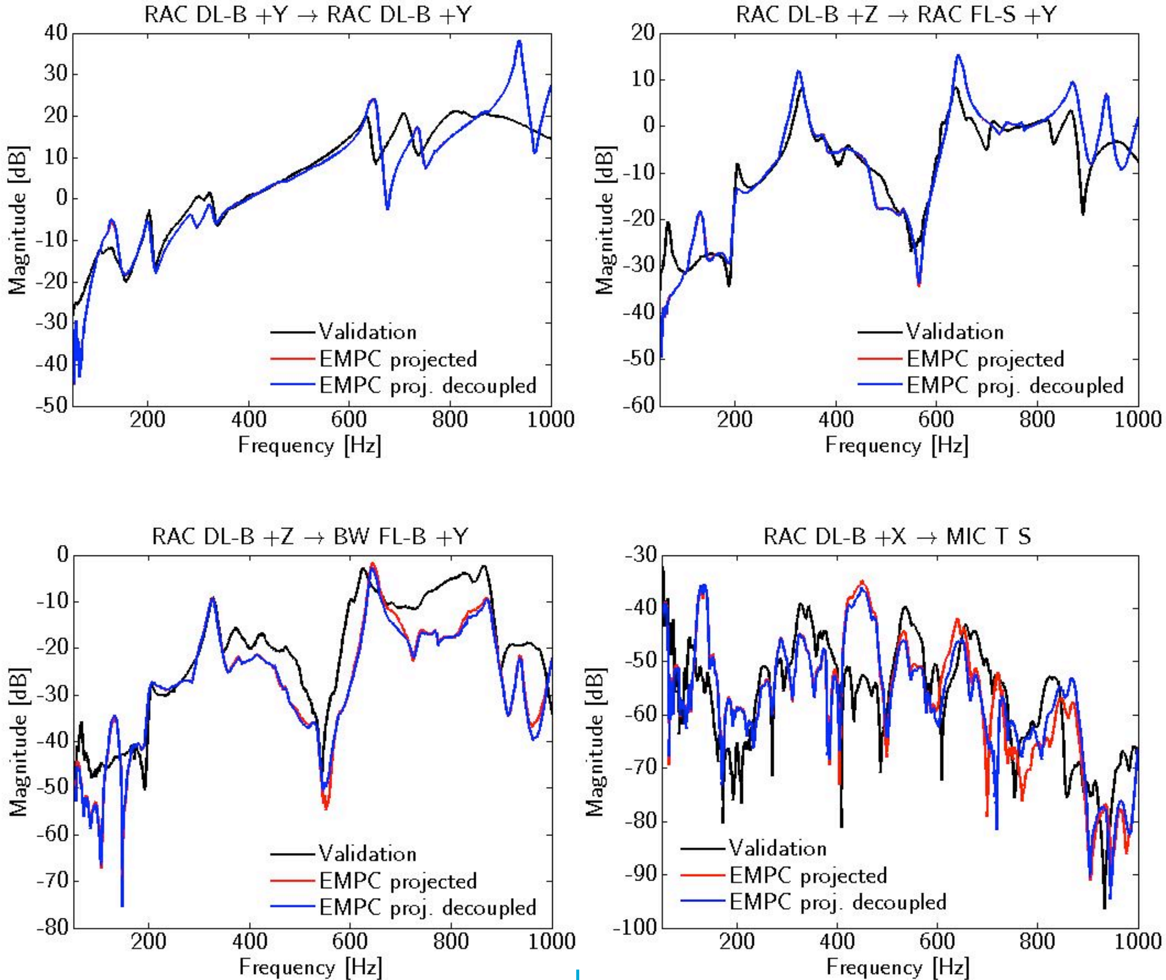
Modeling of Substructures
Body work (BW)



Microphones
Air springs
BW mountings

TUDelft
Delft University of Technology

Example of results



A lot of potential
but …
remaining challenges:

Model reduction and substructuring

- Choose the best substructure reduction automatically
- A priori error estimates (error propagation)
- Reducing parametric models (optimization, updating …)
- Include non-linearities
- Reducing damping matrices

Experimental substructuring

- Measuring rotational dofs
- Obtaining clean FRFs (using a model for instance)
- What if non-linearities
- Apply to impact problems
- Error propagation
- Substructure decoupling